

Forced Convex Mean Curvature Flow in Euclidean Spaces

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Abstract

In this paper, we consider the mean curvature flow of convex hypersurfaces in Euclidean spaces with a general forcing term. We show that the flow may shrink to a point in finite time if the forcing term is small, or exist for all times and expand to infinity if the forcing term is large enough. The flow can also converge to a round sphere for some special forcing term and initial hypersurface. Furthermore, the normalization of the flow is carried out so that long time existence and convergence of the rescaled flow are studied. Our work extends Huisken's well-known mean curvature flow and McCoy's mixed volume preserving mean curvature flow.

1 Introduction

Let M^n be a smooth and compact manifold of dimension $n \geq 2$ without boundary, and $X_0 : M^n \longrightarrow \mathbb{R}^{n+1}$ be a smooth hypersurface immersion of M^n which is strictly convex. We consider a smooth family of maps $X_t = X(\cdot, t)$ evolving according to

$$\begin{cases} \frac{\partial}{\partial t} X(x, t) &= \{h(t) - H(x, t)\} \mathbf{v}(x, t), \quad x \in M^n, \\ X(\cdot, 0) &= X_0, \end{cases} \quad (1.1)$$

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where H is the mean curvature of $M_t = X_t(M^n)$, \mathbf{v} the outer unit normal vector field, and $h(t)$ a nonnegative continuous function. The curvature flow (1.1) is a strictly parabolic equation and the short time existence easily follows from [9]. Therefore we suppose that the evolution equation (1.1) has a smooth solution on a maximal time interval $[0, T_{\max})$ for some $T_{\max} > 0$. Often different forcing term will lead to different maximal time interval. We always assume that $h(t)$ is continuous in $[0, T_{\max})$.

If $h(t) = 0$, (1.1) is just the well-known mean curvature flow [7]. In this case, (1.1) is contracting and T_{\max} is finite. If $h(t)$ is the average of the mean curvature on M_t , i.e. $h(t) = \int_{M_t} H_t d\mu_t / \int_{M_t} d\mu_t$, where $d\mu_t$ is the area element of M_t , (1.1) is then the volume preserving mean curvature flow [8], which exists on all time $[0, \infty)$, and the solution converges to a round sphere. The hypersurfaces area preserving mean curvature flow for which $h(t) = \int_{M_t} H_t^2 d\mu_t / \int_{M_t} H_t d\mu_t$ also exists for all time and converges to a round sphere [12]. The mixed volume preserving mean curvature flow [13] for which $h(t) = \int_{M_t} H E_{k+1} d\mu_t / \int_{M_t} E_{k+1} d\mu_t$, $k = -1, 0, 1, \dots, n-1$, where E_l is the l -th elementary symmetric function of the principal curvatures of M_t , generalizes the results of the volume preserving mean curvature flow [8] and surfaces area preserving mean curvature flow [13], and exists for all time and converges to a round sphere. In fact, it can be checked that if the forcing term h is a small constant, the solution to (1.1) is still contracting. But if h is large enough, the curvature flow (1.1) expands and the solution exists for all time.

From above, we see that different forcing term $h(t)$ leads to different existence and convergence. A natural question is how to unify all these cases?

In this paper, we study the curvature flow (1.1) with a general forcing term $h(t)$ such that the limit $\lim_{t \rightarrow T_{\max}} h(t)$ exists. We want to show that if the initial hypersurface is convex and compact, the shape of M_t approaches the shape of a round sphere as $t \rightarrow T_{\max}$. In order to describe the shape of the limiting hypersurface, we carry out a normalization as in [7]. For any time t , where the solution $X(\cdot, t)$ of (1.1) exists, let $\psi(t)$ be a positive factor such that the hypersurface \widetilde{M}_t given by

$$\widetilde{X}(x, t) = \psi(t)X(x, t)$$

has total area equal to $|M_0|$, the area of M_0

$$\int_{\widetilde{M}_t} d\widetilde{\mu}_t = |M_0|, \quad \text{for all } t \in [0, T_{\max}).$$

After choosing the new time variable $\tilde{t}(t) = \int_0^t \psi^2(\tau) d\tau$, we will see that \widetilde{X} satisfies the following evolution equation

$$\begin{cases} \frac{\partial}{\partial \tilde{t}} \widetilde{X} = \{\tilde{h} - \widetilde{H}\} \widetilde{\mathbf{v}} + \frac{1}{n} \widetilde{\theta} \widetilde{X}, \\ \widetilde{X}(\cdot, 0) = X_0, \end{cases} \quad (1.2)$$

where $\tilde{h} = \psi^{-1}h$, $\tilde{\theta} = \psi^{-2}\theta$ and θ is given by

$$\theta = -\frac{\int_M (h - H)H d\mu}{\int_M d\mu}.$$

In section 3, we have a time sequence $\{T_i\}$ such that $T_i \rightarrow T_{\max}$ as $i \rightarrow \infty$, and a limit

$$\lim_{T_i \rightarrow T_{\max}} \psi(T_i) = \Lambda.$$

We now state our main theorem:

Theorem 1. *Let $n \geq 2$ and M_0 an n -dimensional smooth, compact and strictly convex hypersurface immersed in \mathbb{R}^{n+1} . Then for any nonnegative continuous function $h(t)$, there exists a unique, smooth solution to the evolution equation (1.1) on a maximal time interval $[0, T_{\max})$. If additionally the following limit exists and satisfies*

$$\lim_{t \rightarrow T_{\max}} h(t) = \bar{h} < +\infty, \quad (1.3)$$

then we have:

(I) *If $\Lambda = \infty$, then $T_{\max} < \infty$ and the curvature flow (1.1) converges uniformly to a point as $t \rightarrow T_{\max}$. Moreover the normalized equation (1.2) has a solution $\tilde{X}(x, \tilde{t})$ for all times $0 \leq \tilde{t} < \infty$, and the hypersurfaces $\tilde{M}(x, \tilde{t})$ converge to a round sphere of area $|M_0|$ in the C^∞ -topology, as $\tilde{t} \rightarrow \infty$.*

(II) *If $0 < \Lambda < \infty$, then $T_{\max} = \infty$, and the solutions to (1.1) converge uniformly to a round sphere in the C^∞ -topology as $t \rightarrow \infty$.*

(III) *If $\Lambda = 0$, then $T_{\max} = \infty$. Moreover if $\bar{h} \neq 0$, the solutions to (1.1) expand uniformly to ∞ as $t \rightarrow \infty$ and if the rescaled solutions to (1.2) converge to a smooth hypersurface, then the limit must be a round sphere of total area $|M_0|$.*

Remark 1. (i) *One can check that Theorem 1 includes Huisken's mean curvature flow [7] and volume preserving mean curvature flow [8], McCoy's surface area preserving mean curvature flow [12] and mixed volume preserving mean curvature flow [13].*

(ii) *The assumption (1.3) seems not natural since often the maximal existing time T_{\max} of (1.1) depends on $h(t)$. In fact we can use a stronger assumption that $h(t)$ is a nonnegative continuous function on $[0, \infty)$ and satisfies $\lim_{t \rightarrow \infty} h(t) < +\infty$. Our result still includes all cases in (i).*

The extreme cases of Theorem 1 can also be considered.

Remark 2. (i) For case (I), when $\bar{h} = \infty$, T_{\max} may not be finite, even though M_t is contracting (see Remark 3 (ii) in section 4). A sphere: $r(t) = \frac{1}{t+1}$, $h(t) = n(t+1) - \frac{1}{(t+1)^2}$, is such an example, whose maximal existing time $T_{\max} = \infty$.

(ii) For case (III), if $\bar{h} = 0$, T_{\max} is also infinite (see section 6). We don't know whether the solutions to (1.1) expand uniformly to ∞ as $t \rightarrow \infty$, but we can find the special solution satisfying that condition. In fact, a sphere: $r(t) = \sqrt{t+1}$, $h(t) = \frac{2n+1}{2\sqrt{t+1}}$, is such a particular example, for which M_t expands to infinity. If $\bar{h} = \infty$, by similar discussion as in section 6, we can show that M_t expands to infinity, but T_{\max} may not be ∞ . For example, the sphere $r(t) = \frac{1}{1-t}$, $h(t) = n(1-t) + \frac{1}{(1-t)^2}$ is a solution to (1.1), for which $T_{\max} = 1$, and $r \rightarrow \infty$, as $t \rightarrow 1$.

We remark that Curvature flow in Euclidean spaces with different forcing terms $h(t)$ were also studied by Schnürer-Smoczyk [15], and Liu-Jian [11]. If the ambient space is a Minkowski space, Aarons [1] studied the forced mean curvature flow of graphs and obtained the long time existence and convergence under suitable assumptions on $h(t)$. And a kind of trichotomy to the initial hypersurface was used by Chou-Wang [4] in logarithmic Gauss curvature flow.

This paper is organized as follows: Section 2 introduces some known results on curvature flow (1.1) and some preliminary facts of convex hypersurfaces, which will be used later. In section 3, we carry out the normalization of (1.1), and estimate the inner and outer radii of the rescaled convex hypersurfaces. In terms of the limiting shape of the scaling factor $\psi(t)$ as $t \rightarrow T_{\max}$, long time existence and convergence of solutions to (1.1) or (1.2) are proved in section 4, 5 and 6, separately, and therefore we complete the proof of Theorem 1.

2 Preliminaries

Let M be a smooth hypersurface immersion in \mathbb{R}^{n+1} . We will use the same notation as in [8]. In particular, for a local coordinate system $\{x^1, \dots, x^n\}$ of M , $g = g_{ij}$ and $A = h_{ij}$ denote respectively the metric and second fundamental form of M . Then the mean curvature and the square of the second fundamental form are given by

$$H = g^{ij}h_{ij}, \quad |A|^2 = g^{ij}g^{lm}h_{il}h_{jm},$$

where g^{ij} is the (i, j) -entry of the inverse of the matrix (g_{ij}) . In the sequel we will use λ_i to denote the i -th principle curvature of the hypersurface. Throughout this paper we sum over repeated indices from 1 to n unless otherwise indicated.

The system of (1.1) is a strictly parabolic equation for which short time existence is well known. The gradient on M_t and Beltrami-Laplace operator on

M_t are denoted by ∇ and Δ respectively. As in [8, 13], we have the following evolution equations for various geometric quantities under the flow (1.1)

Lemma 1. *The following evolution equations hold for any solution to equation (1.1)*

- (i) $\frac{\partial}{\partial t} g_{ij} = 2(h - H)h_{ij}.$
- (ii) $\frac{\partial}{\partial t} d\mu_t = H(h - H)d\mu_t.$
- (iii) $\frac{\partial}{\partial t} \mathbf{v} = \nabla H.$
- (iv) $\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} + (h - 2H)h_{ik}h_j^k + |A|^2 h_{ij}.$
- (v) $\frac{\partial}{\partial t} H = \Delta H - (h - H)|A|^2.$
- (vi) $\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2h\text{tr}(A^3).$

Here $d\mu_t$ is the area element of M_t , and $h_i^j = h_{ik}g^{kj}$.

Since M_0 is strictly convex, the curvature flow (1.1) preserves the convexity of all M_t as long as the solution exists [8, 13].

Lemma 2. (i) *If $h_{ij} \geq 0$ at $t = 0$, then it remains so on $[0, T_{\max})$.*

(ii) *If initially $H > 0$ and $h_{ij} \geq \varepsilon H g_{ij}$ for some $\varepsilon \in (0, \frac{1}{n}]$, then $h_{ij} \geq \varepsilon H g_{ij}$ remains true, with the same ε on $[0, T_{\max})$.*

This leads to the following consequence of convexity [7]

Lemma 3. *If initially $H > 0$ and $h_{ij} \geq \varepsilon H g_{ij}$ for some $\varepsilon \in (0, \frac{1}{n}]$ then*

- (i) $H\text{tr}(A^3) - |A|^4 \geq n\varepsilon^2 H^2(|A|^2 - \frac{1}{n}H^2).$
- (ii) $|H\nabla_i h_{kl} - h_{kl}\nabla_i H|^2 \geq \frac{1}{2}\varepsilon^2 H^2 |\nabla H|^2.$

Let $|M|$ be the area of M , and $|V|$ the volume of the region V contained inside M . Lemma 2 implies that every solution of (1.1) is a compact, convex hypersurface, therefore we have the following relations between $|V|$ and $|M|$ by Aleksandrov-Fenchel inequality and divergence theorem (see Theorem 2.3 in [13])

Lemma 4. *Let M be a compact and convex hypersurface embedded into \mathbb{R}^{n+1} satisfying $H > 0$ and $h_{ij} \geq \varepsilon H g_{ij}$, for some $\varepsilon \in (0, \frac{1}{n}]$. Then there exists a constant c_1 depending on n and ε such that*

$$c_1^{-1}|M|^{\frac{n+1}{n}} \leq |V| \leq c_1|M|^{\frac{n+1}{n}}.$$

In order to study (1.1), the following facts of convex hypersurfaces will be used.

Recall that the second fundamental form of a convex hypersurface $X : M^n \rightarrow \mathbb{R}^{n+1}$ is positive definite, and the outer unit normal vector field \mathbf{v} to the hypersurface defines the Gauss map $\mathbf{v} : M^n \rightarrow \mathbb{S}^n$. Since the hypersurface is convex and compact, i.e. the Gauss map is everywhere non-degenerate, we use the Gauss map to reparametrize the convex hypersurface (see [2, 16, 17])

$$X = X(\mathbf{v}^{-1}(z)), \quad z \in \mathbb{S}^n.$$

Then the support function is defined as

$$\mathcal{Z}(z) = \langle z, X(\mathbf{v}^{-1}(z)) \rangle, \quad z \in \mathbb{S}^n.$$

If we denote by $\bar{\nabla}$ and \bar{g} the covariant derivative and standard metric on \mathbb{S}^n , the hypersurface can be represented by the support function

$$X(z) = \mathcal{Z}(z)z + \bar{\nabla}\mathcal{Z}(z).$$

The second fundamental form now can be calculated directly from the support function as follows

$$h_{ij} = \bar{\nabla}_i \bar{\nabla}_j \mathcal{Z} + \mathcal{Z} \bar{g}_{ij} \quad \text{on } \mathbb{S}^n, \quad (2.1)$$

and the metric is given by

$$g_{ij} = h_{ik} \bar{g}^{kl} h_{lj}. \quad (2.2)$$

The width function of the hypersurface X is defined by

$$w(z) = \mathcal{Z}(z) + \mathcal{Z}(-z), \quad z \in \mathbb{S}^n.$$

In order to control the width of a convex hypersurface, we cite a theorem of Andrews [2]

Lemma 5. *Let M be a smooth, compact and convex hypersurface in \mathbb{R}^{n+1} . Suppose that there exists a positive constant c_2 such that M satisfies the pointwise pinching estimate $\lambda_{\max}(x) \leq c_2 \lambda_{\min}(x)$, for every $x \in M$. Then the following estimate holds*

$$w_{\max} \leq c_2 w_{\min},$$

where $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ are the largest and smallest principal curvatures of M at x respectively, and $w_{\max} = \max_{z \in \mathbb{S}^n} w(z)$ and $w_{\min} = \min_{z \in \mathbb{S}^n} w(z)$.

By this lemma, a pinching estimate on the inner radius r_{in} and outer radius r_{out} immediately follows [2]

Corollary 1. *Let M be a smooth, compact and convex hypersurface in \mathbb{R}^{n+1} . Suppose that there exists a positive constant c_2 such that M satisfies the pointwise pinching estimate $\lambda_{\max}(x) \leq c_2 \lambda_{\min}(x)$, for every $x \in M$. Then there exists a constant c_3 such that*

$$r_{out} \leq c_3 r_{in}.$$

For a convex hypersurface M^n , we can also parametrize it as a graph over the unit sphere \mathbb{S}^n (cf. [2, 5], see also [17]). Let

$$\pi(x) = \frac{X(x)}{|X(x)|} : M^n \longrightarrow \mathbb{S}^n,$$

then we write the solution M_t to equation (1.1) as a radial graph

$$X(x, t) = r(z, t)z : \mathbb{S}^n \longrightarrow \mathbb{R}^{n+1}, \quad (2.3)$$

where $r(z, t) = |X(\pi^{-1}(z), t)|$. We calculate the metric of M_t in terms of r as

$$g_{ij} = r^2 \bar{g}_{ij} + \bar{\nabla}_i r \bar{\nabla}_j r,$$

and its inverse is

$$g^{ij} = r^{-2} \left(\bar{g}^{ij} - \frac{\bar{\nabla}^i r \bar{\nabla}^j r}{r^2 + |\bar{\nabla} r|^2} \right). \quad (2.4)$$

The outer unit normal vector and the second fundamental form of M_t in terms of r are given respectively by

$$\mathbf{v} = \frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} (rz - \bar{\nabla} r), \quad (2.5)$$

and

$$h_{ij} = \frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} (-r \bar{\nabla}_i \bar{\nabla}_j r + 2 \bar{\nabla}_i r \bar{\nabla}_j r + r^2 \bar{g}_{ij}). \quad (2.6)$$

3 The Normalized Equation

The solution of the curvature flow (1.1) may shrink to a point if h is small enough (e.g. $h = 0$ [7]), or expand to infinity if h is large enough (e.g. h is a constant and $h > \sup_{x \in M^n} H(x, 0)$). The solution can also converge to a smooth hypersurface, for some special initial hypersurface and h (e.g. the volume preserving mean curvature flow [8], the surface area preserving mean curvature flow [12]). In order to see this, we normalize the equation (1.1) by keeping some geometrical quantity fixed, for example as in [7] the total area of

the hypersurfaces M_t . As that mentioned in section 1, multiplying the solution X of (1.1) at each time $0 \leq t < T_{\max}$ with a positive constant $\psi(t)$ such that the total area of the hypersurfaces \widetilde{M}_t given by

$$\widetilde{X}(x, t) = \psi(t)X(x, t)$$

has total area equal to $|M_0|$, the area of M_0

$$\int_{\widetilde{M}_t} d\widetilde{\mu}_t = |M_0|, \quad 0 \leq t < T_{\max}. \quad (3.1)$$

Then we introduce a new time variable $\tilde{t}(t) = \int_0^t \psi^2(\tau) d\tau$, such that $\frac{\partial \tilde{t}}{\partial t} = \psi^2$.

As in [7, 2], for a geometric quantity P on M_t , we denote by \widetilde{P} the corresponding quantity on the rescaled hypersurface $\widetilde{M}_{\tilde{t}}$. By direct calculation we have

$$\begin{aligned} \widetilde{g}_{ij} &= \psi^2 g_{ij}, & \widetilde{h}_{ij} &= \psi h_{ij}, \\ \widetilde{H} &= \psi^{-1} H, & |\widetilde{A}|^2 &= \psi^{-2} |A|^2, \\ d\widetilde{\mu} &= \psi^n d\mu, & \widetilde{w} &= \psi w, \end{aligned}$$

and so on. If we differentiate (3.1) for time t , we obtain

$$\psi^{-1} \frac{\partial \psi}{\partial t} = \frac{1}{n} \frac{\int_M (H - h) H d\mu}{\int_M d\mu} = \frac{1}{n} \theta.$$

Now by differentiating \widetilde{X} with respect to \tilde{t} , we derive the normalized evolution equation for a different maximal time interval $0 \leq \tilde{t} < \widetilde{T}_{\max}$

$$\begin{cases} \frac{\partial}{\partial \tilde{t}} \widetilde{X}(x, \tilde{t}) = \{\widetilde{h}(\tilde{t}) - \widetilde{H}(x, \tilde{t})\} \widetilde{\nabla}(x, \tilde{t}) + \frac{1}{n} \widetilde{\theta}(\tilde{t}) \widetilde{X}(x, \tilde{t}), \\ \widetilde{X}(\cdot, 0) = X_0, \end{cases} \quad (3.2)$$

where $\widetilde{h} = \psi^{-1} h$, $\widetilde{\theta} = \psi^{-2} \theta$ and θ is given by

$$\theta = - \frac{\int_M (h - H) H d\mu}{\int_M d\mu}.$$

Since M_t is convex, and $\widetilde{M}_{\tilde{t}}$ is just a rescaling of M_t , therefore which is also convex, we can write M_t or $\widetilde{M}_{\tilde{t}}$ to be a graph over a unit sphere as in (2.3). By (1.1), (2.4)~(2.6) we have the evolution equation for $r(t)$

$$\frac{\partial r}{\partial t} = \frac{h}{r} \sqrt{r^2 + |\widetilde{\nabla} r|^2} + r^{-3} \left(\widetilde{g}^{ij} - \frac{\widetilde{\nabla}^i r \widetilde{\nabla}^j r}{r^2 + |\widetilde{\nabla} r|^2} \right) (r \widetilde{\nabla}_i \widetilde{\nabla}_j r - 2 \widetilde{\nabla}_i r \widetilde{\nabla}_j r - r^2 \widetilde{g}_{ij}). \quad (3.3)$$

Then $\tilde{r} = \psi r$ satisfies the evolution equation

$$\begin{aligned} \frac{\partial \tilde{r}}{\partial t} &= \frac{\tilde{\theta}}{n} \tilde{r} + \frac{\tilde{h}}{\tilde{r}} \sqrt{\tilde{r}^2 + |\nabla \tilde{r}|^2} \\ &\quad + \tilde{r}^{-3} \left(\tilde{g}^{ij} - \frac{\nabla^i \tilde{r} \nabla^j \tilde{r}}{\tilde{r}^2 + |\nabla \tilde{r}|^2} \right) (\tilde{r} \nabla_i \nabla_j \tilde{r} - 2 \nabla_i \tilde{r} \nabla_j \tilde{r} - \tilde{r}^2 \tilde{g}_{ij}). \end{aligned} \quad (3.4)$$

In the remainder of this section, we will estimate the outer and inner radii of the normalized hypersurfaces \widetilde{M} . First we see that since at each time the whole configuration of \widetilde{M} is only dilated by a constant factor ψ , the solutions to (3.2) are compact and convex hypersurfaces, and Lemma 2 still holds. This means that

$$\tilde{h}_{ij} \geq \varepsilon \tilde{H} \tilde{g}_{ij},$$

for some $\varepsilon \in (0, \frac{1}{n}]$. The hypersurface \widetilde{M} encloses a region \widetilde{V} of volume $|\widetilde{V}|$. Then by Lemma 4

$$c_1^{-1} |\widetilde{M}|^{\frac{n+1}{n}} \leq |\widetilde{V}| \leq c_1 |\widetilde{M}|^{\frac{n+1}{n}}. \quad (3.5)$$

Since $|\widetilde{V}|$ is controlled by the volume of its inner and outer sphere

$$c_4 \tilde{r}_{in}^{n+1} \leq |\widetilde{V}| \leq c_4 \tilde{r}_{out}^{n+1},$$

for a constant c_4 , we obtain the following estimate by the fixed total area of \widetilde{M} by (3.5)

$$\tilde{r}_{out} \geq c_5 \text{ and } \tilde{r}_{in} \leq c_6, \quad (3.6)$$

for some two positive constants c_5 and c_6 .

By Corollary 1 and (3.6) we have

Proposition 1. *The lower bound of the inner radius and the upper bound of the outer radius of $\widetilde{M}_{\tilde{t}}$ are all uniformly bounded, i.e.*

$$c_7^{-1} \leq \tilde{r}_{in} \leq \tilde{r}_{out} \leq c_7$$

for some constant c_7 .

Now for any given time sequence $\{T_i\}$, $T_i \in [0, T_{\max})$, such that $T_i \rightarrow T_{\max}$ as $i \rightarrow \infty$, there corresponds to a sequence $\{\psi_i = \psi(T_i)\}$. By limiting theory, there exists at least one accumulation of this sequence. Denote by Λ_i the minimal accumulation of the sequence $\{\psi_i = \psi(T_i)\}$. We define Λ to be the infimum of Λ_i for all possible sequences $\{\psi_i = \psi(T_i)\}$, i.e.

$$\begin{aligned} \Lambda &= \inf \{ \Lambda_i | \Lambda_i \text{ is the minimal accumulation of a sequence } \{\psi_i = \psi(T_i)\} , \\ &\quad \text{where } \{T_i\} \text{ is any sequence in } [0, T_{\max}) \text{ such that } T_i \rightarrow T_{\max} \text{ as } i \rightarrow \infty \}. \end{aligned}$$

Therefore by the method of extracting diagonal subsequences we have a subsequence, still denoted by $\{\psi_i = \psi(T_i)\}$, which converges to Λ as $T_i \rightarrow T_{\max}$ (or $i \rightarrow \infty$), that is to say we have the following limit

$$\lim_{i \rightarrow \infty} \psi_i = \Lambda. \quad (3.7)$$

There are three cases in terms of the limit Λ : $\Lambda = \infty$, $0 < \Lambda < \infty$ and $\Lambda = 0$. We will consider the three cases separately in the sequel.

4 Case (I) $\Lambda = \infty$

In this section we consider the case $\Lambda = \infty$, and prove Theorem 1(I). Since $\tilde{r}_{out} = r_{out}\psi$, we have by Proposition 1

$$\frac{c_7^{-1}}{\psi} \leq r_{out} \leq \frac{c_7}{\psi},$$

which implies that for the sequence $\{T_i\}$ in last section (see (3.7)), we have a limit

$$\lim_{T_i \rightarrow T_{\max}} r_{out}(T_i) = 0. \quad (4.1)$$

By limiting theory, there exists a time $T^* < T_{\max}$ such that for any $T_i \geq T^*$, $r_{out}(T_i)$ is less than any given positive number r^* . By the assumption (1.3), $h(t)$ has a uniformly upper bound h^+ on $[0, T_{\max})$ (We can always assume $h^+ > 0$ even in the case of mean curvature flow, i.e. $h(t) = 0$). We now choose r^* is less than n/h^+ .

We follow an idea in [2, 17] to prove the following lemma which implies that when t is very near T_{\max} , M_t is in fact contracting.

Lemma 6. *When $t \geq T^*$, the regions enclosed by the hypersurfaces M_t are decreasing. Furthermore $T_{\max} < \infty$, and the solutions to (1.1) converge uniformly to a point in \mathbb{R}^{n+1} as $t \rightarrow T_{\max}$.*

Proof. Let $\partial B_{r^*}(O)$ be a sphere in \mathbb{R}^{n+1} centered at the origin O , with radius r^* . Since the outer radius of M_{T^*} is less than r^* , without loss of generality, we may assume that the hypersurface M_{T^*} is enclosed by $\partial B_{r^*}(O)$. Now we evolve the sphere $\partial B_{r^*}(O)$ in terms of (1.1), the radius $r_B(t)$ satisfies

$$\begin{cases} \frac{dr_B(t)}{dt} = h - \frac{n}{r_B(t)} \leq h^+ - \frac{n}{r_B(t)}, & t \geq T^*, \\ r_B(T^*) = r^*, \end{cases} \quad (4.2)$$

which yields that $r_B(t)$ is decreasing because $r^* < n/h^+$. Then by containment principle, which can be easily derived from (3.3), we see that the enclosed regions of M_t are decreasing for $t \geq T^*$.

Furthermore it can be checked that the solution to the differential inequality (4.2) is given by

$$r_B(t) + \frac{n}{h^+} \log(n - h^+ r_B(t)) \geq h^+(t - T^*) + r^* + \frac{n}{h^+} \log(n - h^+ r^*), \quad (4.3)$$

which yields the finiteness of T_{\max} since the left hand side of (4.3) is uniformly bounded for $t \geq T^*$.

By convexity in Lemma 2, the pinching estimate in Corollary 1 will imply the uniform convergence of solutions to (1.1) to a point if we can show that the enclosed area of M_t tends to 0 as $t \rightarrow T_{\max}$. If this is not true, we then can place a small ball $B_{r_0}(x_0)$ in the region enclosed by M_t for all $t \in [T^*, T_{\max})$. Again without loss of generality we assume x_0 is the origin. Then the diameter of M_t is uniformly bounded from below, and $|\nabla r|$ is also uniformly bounded by convexity. Therefore equation (3.3) is a uniformly parabolic equation with bounded coefficients. Hence we can apply the standard regularity theory of uniformly parabolic equations (cf. [10] or [2, 17]) to conclude that the solution to (3.3) can not be singular at $t = T_{\max}$, which is a contradiction. Therefore $X(\cdot, t)$ must converge to a point as $t \rightarrow T_{\max}$. This completes the proof of the lemma. \square

Remark 3. (i) From the proof of Lemma 6, we see that the containment principle implies that r_{out} tends to zero, as $t \rightarrow T_{\max}$. Therefore by Proposition 1 again, the function $\psi(t)$ must tend to infinity as $t \rightarrow T_{\max}$, i.e.

$$\lim_{t \rightarrow T_{\max}} \psi(t) = \infty. \quad (4.4)$$

(ii) We can see that for $\bar{h} = \infty$, (1.1) is still contracting to a point. In fact from the limit of $\psi(T_i)$ in section 3, we see that Λ is the smallest limit of ψ . That is to say if $\Lambda = \infty$, then for any sequence $\{T_j\} \subset [0, T_{\max})$ satisfying $T_j \rightarrow T_{\max}$ as $j \rightarrow \infty$, $\lim_{j \rightarrow \infty} \psi(T_j) = \infty$. Therefore similarly by Proposition 1, the inner and outer radii of the evolving hypersurfaces all tend to zero as $t \rightarrow T_{\max}$. Then the containment principle implies that the solutions to (1.1) converge to a point as $t \rightarrow T_{\max}$ for all possible limits of $h(t)$.

To understand the solution $X(\cdot, t)$ near the maximal time T_{\max} , we consider the solution of the rescaled equation (3.2). We want to bound the curvature \tilde{H} of \tilde{M}_t , for this purpose, we will use a trick of Chow (Tso) [14] (see also [2, 13, 17]) to consider the function

$$\Phi = \frac{H}{\mathcal{Z} - \alpha}, \quad (4.5)$$

for a constant α to be chosen later. First we compute the evolution equation of Φ .

Lemma 7. *For $t \in [0, T_{\max})$, for any constant α we have*

$$\begin{aligned} \frac{\partial}{\partial t} \Phi &= g^{ij} \bar{\nabla}_i \bar{\nabla}_j \Phi + \frac{2}{\mathcal{Z} - \alpha} g^{ij} \bar{\nabla}_i \Phi \bar{\nabla}_j \mathcal{Z} \\ &\quad + \frac{1}{(\mathcal{Z} - \alpha)^2} \{2H^2 - hH - \alpha H |A|^2 - h(\mathcal{Z} - \alpha) |A|^2\}. \end{aligned} \quad (4.6)$$

Proof. The proof is just the one in [13]. Because we shall consider the evolution equations of similar functions in section 5 and 6, we outline its proof here. We first have

$$\bar{\nabla}_i \Phi = \frac{\bar{\nabla}_i H}{\mathcal{Z} - \alpha} - \frac{H \bar{\nabla}_i \mathcal{Z}}{(\mathcal{Z} - \alpha)^2},$$

and

$$\bar{\nabla}_i \bar{\nabla}_j \Phi = \frac{\bar{\nabla}_i \bar{\nabla}_j H}{\mathcal{Z} - \alpha} - \frac{\bar{\nabla}_i H \bar{\nabla}_j \mathcal{Z} + \bar{\nabla}_i \mathcal{Z} \bar{\nabla}_j H}{(\mathcal{Z} - \alpha)^2} - \frac{H \bar{\nabla}_i \bar{\nabla}_j \mathcal{Z}}{(\mathcal{Z} - \alpha)^2} + \frac{2H \bar{\nabla}_i \mathcal{Z} \bar{\nabla}_j \mathcal{Z}}{(\mathcal{Z} - \alpha)^3},$$

which yields

$$g^{ij} \bar{\nabla}_i \bar{\nabla}_j \Phi = \frac{g^{ij} \bar{\nabla}_i \bar{\nabla}_j H}{\mathcal{Z} - \alpha} - \frac{2g^{ij} \bar{\nabla}_i \Phi \bar{\nabla}_j \mathcal{Z}}{\mathcal{Z} - \alpha} - \frac{H g^{ij} \bar{\nabla}_i \bar{\nabla}_j \mathcal{Z}}{(\mathcal{Z} - \alpha)^2}. \quad (4.7)$$

By differentiating the support function with respect to time t we have

$$\frac{\partial \mathcal{Z}}{\partial t} = h - H.$$

By using (2.2), one has

$$H = g^{ij} h_{ij} = \bar{g}_{ij} (h^{-1})^{ij},$$

where $(h^{-1})^{ij}$ is the inverse of h_{ij} . Thus by (2.1) we have the evolution equation of H in terms of the connection on \mathbb{S}^n

$$\frac{\partial H}{\partial t} = g^{ij} [\bar{\nabla}_i \bar{\nabla}_j H + (H - h) \bar{g}_{ij}].$$

Then the time derivative of Φ is given by

$$\frac{\partial \Phi}{\partial t} = \frac{g^{ij}}{\mathcal{Z} - \alpha} [\bar{\nabla}_i \bar{\nabla}_j H + (H - h) \bar{g}_{ij}] - \frac{H(h - H)}{(\mathcal{Z} - \alpha)^2}. \quad (4.8)$$

Now by (2.2) again, we have the identity $g^{ij}\bar{g}_{ij} = |A|^2$. Therefore by combining (4.7) and (4.8), we obtain the expression

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= g^{ij}\bar{\nabla}_i\bar{\nabla}_j\Phi + \frac{2}{\mathcal{Z}-\alpha}g^{ij}\bar{\nabla}_i\Phi\bar{\nabla}_j\mathcal{Z} \\ &\quad + \frac{Hg^{ij}\bar{\nabla}_i\bar{\nabla}_j\mathcal{Z}}{(\mathcal{Z}-\alpha)^2} - \frac{h-H}{\mathcal{Z}-\alpha}|A|^2 - \frac{H(h-H)}{(\mathcal{Z}-\alpha)^2} \\ &= g^{ij}\bar{\nabla}_i\bar{\nabla}_j\Phi + \frac{2}{\mathcal{Z}-\alpha}g^{ij}\bar{\nabla}_i\Phi\bar{\nabla}_j\mathcal{Z} \\ &\quad + \frac{1}{(\mathcal{Z}-\alpha)^2} \{2H^2 - hH - \alpha H|A|^2 - h(\mathcal{Z}-\alpha)|A|^2\}, \end{aligned}$$

which establishes the lemma. \square

For $t \in [0, T^*]$, M_t is smooth, compact and convex, and therefore the mean curvature H is uniformly bounded in this time interval. Similarly, the mean curvature of \widetilde{M} is also bounded in the corresponding time interval. Moreover we can prove the following

Lemma 8. *There exists a positive constant c_8 such that for any $\tilde{t} \in [0, \tilde{T}_{\max})$,*

$$\widetilde{H}(x, \tilde{t}) \leq c_8, \quad \forall x \in M^n.$$

Proof. Let $\tilde{T}^* = \int_0^{T^*} \psi^2(t)dt$. For any $\tilde{t} \in [0, \tilde{T}^*]$, $\widetilde{M}_{\tilde{t}}$ is a smooth, compact and convex hypersurface, the mean curvature \widetilde{H} is therefore uniformly bounded in $[0, \tilde{T}^*]$.

Consider any time $t_0 \in [T^*, T_{\max})$, and choose the origin of \mathbb{R}^{n+1} to be the center of the sphere of radius $r_{in}(t_0)$, which is enclosed by $X(\cdot, t_0)$. By Lemma 6, on the time interval $[T^*, t_0]$, the support function satisfies

$$\mathcal{Z} = \langle X, \mathbf{v} \rangle \geq r_{in}(t_0).$$

Let $\alpha = \frac{1}{2}r_{in}(t_0)$, we consider the function $\Phi(z, t)$ defined in (4.5) for any $(z, t) \in \mathbb{S}^n \times [T^*, t_0]$. Let $(z_1, t_1) \in \mathbb{S}^n \times [T^*, t_0]$ be such that Φ achieves the maximum $\sup\{\Phi(z, t) | (z, t) \in \mathbb{S}^n \times [T^*, t_0]\}$. If $t_1 = T^*$, we are done, since in this case, $H(z, t_0) \leq \text{constant}$. Thus we may assume $t_1 > T^*$, then by Lemma 7, at (z_1, t_1)

$$2H^2 - hH - \alpha H|A|^2 - h(\mathcal{Z} - \alpha)|A|^2 \geq 0.$$

We use $|A|^2 \geq \frac{1}{n}H^2$ and $\mathcal{Z} \geq 2\alpha$ to obtain

$$H(z_1, t_1) \leq \frac{2n}{\alpha}.$$

Therefore for any $z \in \mathbb{S}^n$,

$$\Phi(z, t_0) = \frac{H(z, t_0)}{\mathcal{Z}(z, t_0) - \alpha} \leq \Phi(z_1, t_1),$$

which implies

$$H(z, t_0) \leq \frac{c_9}{r_{in}(t_0)},$$

for a constant c_9 , where we have used Corollary 1. By combining with Proposition 1, we have

$$\tilde{H}(z, \tilde{t}_0) \leq c_{10},$$

for all $z \in \mathbb{S}^n$. Here $\tilde{t}_0 = \int_0^{t_0} \psi^2(t) dt$.

Since $t_0 \in [T^*, T_{\max})$ is arbitrary, $\tilde{t}_0 \in [\tilde{T}^*, \tilde{T}_{\max})$ is also arbitrary, we thus have the uniform bound on \tilde{H} in $[\tilde{T}^*, \tilde{T}_{\max})$. Combination with the bound in $[0, \tilde{T}^*]$, we at last arrive at the inequality $\tilde{H}(x, \tilde{t}) \leq c_8$, for a constant c_8 . \square

We can now prove the following long time existence of (3.2). In section 3, we have bounded the inner radius and the outer radius for $\tilde{X}(\cdot, \tilde{t})$, and in above, we have bounded the speed of the equation (3.2). Thus there is a positive constant $\delta > 0$ such that for each $\tilde{t}_0 \in [0, \tilde{T}_{\max})$, we can write the solution $\tilde{X}(\cdot, \tilde{t})$ to (3.2) on the time interval $[\tilde{t}_0, \tilde{t}_0 + \delta]$ as a graph for some $\delta > 0$

$$\tilde{X}(z, \tilde{t}) = \tilde{r}(z, \tilde{t})z, \quad z \in \mathbb{S}^n$$

for some chosen origin, and satisfies $0 < c_7^{-1} \leq \tilde{r}(z, \tilde{t}) \leq c_7$, on $\mathbb{S}^n \times [\tilde{t}_0, \tilde{t}_0 + \delta]$. By the convexity of all evolving hypersurfaces, we know that $\bar{\nabla} \tilde{r}$ is also uniformly bounded. We write down the evolution equation of \tilde{r} , similar to (3.4), we know that it is uniformly parabolic. So we can use the standard regularity theory of uniformly parabolic equations to bound the derivatives and all higher order derivatives of \tilde{r} (see [10] or [2, 17]). Hence we have proved

Lemma 9. $\tilde{T}_{\max} = \infty$, and $\tilde{M}_{\tilde{t}}$ converges to a smooth hypersurface \tilde{M}_{∞} , as $\tilde{t} \rightarrow \infty$.

Remark 4. By convexity the zero order estimate of \tilde{A} follows from Lemma 8, then one can use the induction argument as in [6] and [7, 8] to show that the curvature derivatives $|\tilde{\nabla}^m \tilde{A}|^2$ are each bounded by a corresponding constant $C_m(n, M_0)$ for any $m \geq 1$, since the terms containing h in the evolution equation can be easily controlled. This in turn can also imply the long time existence of (3.2).

It remains to show that the limiting hypersurface \widetilde{M}_∞ is a round sphere. For this purpose, we define a function

$$\tilde{f} = \frac{|\tilde{A}|^2}{\tilde{H}^2}.$$

It is easy to see that \tilde{f} is a scaling invariant and we have the following lemma similar as in ([13])

Lemma 10. *We have the following evolution equation*

$$\begin{aligned} \frac{\partial}{\partial \tilde{t}} \tilde{f} &= \tilde{\Delta} \tilde{f} + \frac{2}{\tilde{H}} \langle \tilde{\nabla}_l \tilde{f}, \tilde{\nabla}_l \tilde{H} \rangle \\ &\quad - \frac{2}{\tilde{H}^4} |\tilde{H} \tilde{\nabla}_l \tilde{h}_{ij} - \tilde{h}_{ij} \tilde{\nabla}_l \tilde{H}|^2 - \frac{2\tilde{h}}{\tilde{H}^3} (\tilde{H} \text{tr}(\tilde{A}^3) - |\tilde{A}|^4). \end{aligned} \quad (4.9)$$

Proof. First we have the evolution equation of $f = \frac{|A|^2}{H^2}$ (cf. [13])

$$\frac{\partial}{\partial t} f = \Delta f + \frac{2}{H} \langle \nabla_l f, \nabla_l H \rangle - \frac{2}{H^4} |H \nabla_l h_{ij} - h_{ij} \nabla_l H|^2 - \frac{2h}{H^3} (H \text{tr}(A^3) - |A|^4). \quad (4.10)$$

Therefore we have

$$\begin{aligned} \frac{\partial}{\partial \tilde{t}} \tilde{f} &= \frac{\partial}{\partial t} \left(\frac{|A|^2}{H^2} \right) \cdot \frac{\partial t}{\partial \tilde{t}} \\ &= \left\{ \Delta \left(\frac{|A|^2}{H^2} \right) + \frac{2}{H} \langle \nabla_l \left(\frac{|A|^2}{H^2} \right), \nabla_l H \rangle \right. \\ &\quad \left. - \frac{2}{H^4} |H \nabla_l h_{ij} - h_{ij} \nabla_l H|^2 - \frac{2h}{H^3} (H \text{tr}(A^3) - |A|^4) \right\} \cdot \psi^{-2}, \end{aligned}$$

which implies the desired equality. \square

We then can prove the first part of Theorem 1.

Proof. Recalling Lemma 3 we have by Lemma 10,

$$\left(\frac{\partial}{\partial \tilde{t}} - \tilde{\Delta} \right) \tilde{f} \leq \frac{2}{\tilde{H}} \langle \tilde{\nabla}_l \tilde{f}, \tilde{\nabla}_l \tilde{H} \rangle.$$

By the weak maximum principle,

$$\max_{\widetilde{M}_{\tilde{t}}} \tilde{f} \leq \max_{\widetilde{M}_0} \tilde{f}.$$

Furthermore, by the strong maximum principle, if the maximum is attained at some (x, \tilde{t}_0) , $\tilde{t}_0 > 0$, then \tilde{f} is identically constant. Substituting into (4.9) yields

$$\frac{2}{\tilde{H}^4} |\tilde{H} \tilde{\nabla}_l \tilde{h}_{ij} - \tilde{h}_{ij} \tilde{\nabla}_l \tilde{H}|^2 + \frac{2\tilde{h}}{\tilde{H}^3} (\tilde{H} \text{tr}(\tilde{A}^3) - |\tilde{A}|^4) \equiv 0.$$

Now, $\tilde{H} \text{tr}(\tilde{A}^3) - |\tilde{A}|^4 \equiv 0$ implies by Lemma 3 that

$$|\tilde{A}|^2 - \frac{1}{n} \tilde{H}^2 \equiv 0,$$

i.e.

$$\sum_{i < j} (\tilde{\lambda}_i - \tilde{\lambda}_j)^2 \equiv 0,$$

so at any point of $\tilde{M}_{\tilde{t}}$, all the principal curvatures are equal. Also $|\tilde{H} \tilde{\nabla}_l \tilde{h}_{ij} - \tilde{h}_{ij} \tilde{\nabla}_l \tilde{H}|^2 \equiv 0$ implies $\tilde{\nabla} \tilde{H} \equiv 0$ by Lemma 3 (ii), which then implies $\tilde{\nabla} \tilde{A} \equiv 0$, so $\tilde{M}_{\tilde{t}_0}$ is a sphere. Therefore we have showed that the function $\max_{\tilde{M}_{\tilde{t}}} \tilde{f}$ is strictly

decreasing unless $\tilde{M}_{\tilde{t}}$ is a sphere. This implies that $\tilde{M}_{\tilde{t}}$ approaches a sphere as $\tilde{t} \rightarrow \infty$. Of course \tilde{M}_{∞} has the same total area $|M_0|$. Therefore the proof of Theorem 1(I) is completed. \square

Remark 5. (i) One can use a similar method as in [2, 7] to prove that $\tilde{M}_{\tilde{t}}$ converges to a sphere exponentially.

(ii) It is easy to check that $0 \leq h < \inf_{x \in M^n} H(x, 0)$ is of this case, and T^* below (4.1) is equal to zero.

5 Case (II) $0 < \Lambda < \infty$

In this section we consider the case $0 < \Lambda < \infty$ and prove the main Theorem 1(II). Since $\tilde{r}_{out} = r_{out}\psi$ and $\tilde{r}_{in} = r_{in}\psi$, we have by Proposition 1

$$\frac{c_7^{-1}}{\psi} \leq r_{in} \leq r_{out} \leq \frac{c_7}{\psi},$$

which implies for the sequence $\{T_i\}$ in section 3, there exists a time $T^* < T_{\max}$ such that for any $T_i \geq T^*$,

$$c_{12}^{-1} \leq r_{in}(T_i) \leq r_{out}(T_i) \leq c_{12} \quad (5.1)$$

for some constant c_{12} . The following lemma shows that the inner and outer radii of all evolving hypersurfaces M_t are uniformly bounded from below and above.

Lemma 11. *There exists a constant c_{13} such that*

$$c_{13}^{-1} \leq r_{in}(t) \leq r_{out}(t) \leq c_{13}, \quad \text{for any } t \in [0, T_{\max}).$$

Proof. We only prove the upper bound, the lower bound is similar. First we claim that $\bar{h} > 0$ in this case, where \bar{h} is the limit in (1.3). Suppose not, we can take any $h^+ > 0$, such that there exists a time $T' < T_{\max}$ and $h(t) < h^+$ for any $t \in [T', T_{\max})$. Then by similar proof as in Lemma 6, we prove that M_t is contracting for $t \geq T'$. Therefore $r_{out}(T_i) \rightarrow 0$ as $T_i \rightarrow T_{\max}$, which is a contradiction to (5.1). The claim follows.

From the claim we know that there must exist a time $T' \in (T^*, T_{\max})$ such that for any $t \in [T', T_{\max})$, $h(t)$ has a positive lower bound $h^- > 0$.

Since M_t for any $t \in [0, T']$ is smooth, compact and convex, the corresponding outer radius is uniformly bounded from above in this time interval. Suppose there is a time $T'' > T'$ such that $r_{out}(T'') > c_{13}$. By Corollary 1 we can assume c_{13} is large enough so that $r_{in}(T'') > \frac{n}{h^-}$. Again, we evolve a sphere $\partial B_{r_{in}(T'')}(O)$ under (1.1). The solution $r_B(t)$ to the differential inequality

$$\begin{cases} \frac{dr_B(t)}{dt} = h - \frac{n}{r_B(t)} \geq h^- - \frac{n}{r_B(t)}, & t \geq T'', \\ r_B(T'') = r_{in}(T'') > \frac{n}{h^-}, \end{cases}$$

is given by

$$\begin{aligned} r_B(t) + \frac{n}{h^-} \log(h^- r_B(t) - n) &\geq h^-(t - T'') + r_{in}(T'') \\ &\quad + \frac{n}{h^-} \log(h^- r_{in}(T'') - n). \end{aligned}$$

Clearly $r_B(t) \rightarrow \infty$ as $t \rightarrow \infty$. On the other hand, by containment principle, $\partial B_{r_B(t)}(O)$ is enclosed by M_t for any $t \geq T''$, since $M_{T''}$ encloses $\partial B_{r_{in}(T'')}(O)$. Therefore there exists some $T_i > T''$ such that $r_{out}(T_i) \geq r_B(T_i) > c_{12}$, which is a contradiction to (5.1). Combining the case in $[0, T']$, we finish the proof of the lemma. \square

Remark 6. *Similar as in Remark 3, by Lemma 11 and that the hypersurface M_t uniformly converges to a round sphere (see below for the proof), we have a limit*

$$\lim_{t \rightarrow T_{\max}} \psi(t) = \Lambda. \quad (5.2)$$

Based on a theorem of Chow and Gulliver [3], we have as in [12, 13] by Lemma 11 and 4,

Lemma 12. *There is a $d = d(M_0)$ such that $M_t \subset B_d(O)$ for all $t \in [0, T_{\max})$.*

The following lemma also follows from McCoy [13]

Lemma 13. *If $B_{4\alpha}(p_0) \subset V_{t_0}$ for some $t_0 \in [0, T_{\max})$ and a point $p_0 \in \mathbb{R}^{n+1}$, then $B_{2\alpha}(p_0) \subset V_t$ for any $t \in [t_0, t_0 + \min(\frac{6\alpha^2}{n}, T_{\max}))$.*

Similar as in section 4, we consider the function Φ defined in (4.5) for $t \in [t_0, t_0 + \min(\frac{6\alpha^2}{n}, T_{\max}))$, and $\alpha = \frac{1}{4}c_{13}^{-1}$, where c_{13} is given in Lemma 11. By using the same method as in [13], we obtain the uniform upper bound of the evolving mean curvature H .

Lemma 14. *There exists a constant c_{14} such that for any $t \in [0, T_{\max})$*

$$H(x, t) \leq c_{14}, \quad \forall x \in M^n.$$

Again by the standard regular theory of parabolic equations as in section 4, or the argument as in [8, 12, 13], we have

Lemma 15. *$T_{\max} = \infty$, and M_t converges to a smooth hypersurface M_∞ , as $t \rightarrow \infty$.*

Now we can prove the second part of Theorem 1.

Proof. We again consider the function $f = \frac{|A|^2}{H^2}$. By the evolution equation of f in (4.10) and Lemma 3, similar to the proof of Theorem 1(I), we have that $\max_{M_t} f$ is strictly decreasing unless M_t is a sphere. This finishes the proof of Theorem 1(II). \square

Remark 7. (i) *One can also prove that M_t converges to a sphere exponentially as in [8, 12].*

(ii) *By the limit (5.2), we easily see that $\widetilde{M}_{\widetilde{t}}$ converges to a sphere of total area $|M_0|$.*

6 Case (III) $\Lambda = 0$

This section is devoted to discuss the case $\Lambda = 0$ and prove the main Theorem 1(III). Similar to section 4, we have a limit

$$\lim_{T_i \rightarrow T_{\max}} r_{in}(T_i) = \infty. \quad (6.1)$$

Then there exists a time $T^* < T_{\max}$ such that for any $T_i \geq T^*$, $r_{in}(T_i)$ is greater than any given positive number N . As before we evolve $\partial B_{r_{in}(T^*)}(O)$

and $\partial B_{r_{out}(T^*)}(O)$ under (1.1), respectively. That is to say, they satisfy the following equation

$$\frac{dr_B(t)}{dt} = h(t) - \frac{n}{r_B(t)}, \quad t \geq T^*, \quad (6.2)$$

with initial data $r_{in}(T^*)$ and $r_{out}(T^*)$ respectively.

First we consider the case $\bar{h} = 0$. Integrating (6.2) from T^* to T_i and using integral mean-value theorem, the outer radius $r_B^+(t)$ of M_t satisfies

$$r_B^+(T_i) - r_{out}(T^*) = \left[h(t_i) - \frac{n}{r_B^+(t_i)} \right] (T_i - T^*), \quad (6.3)$$

where $t_i \in [T^*, T_i]$.

If we suppose $T_{\max} < \infty$, and take limits of both sides in (6.3), we have $\lim_{t \rightarrow T_{\max}} h(t) = \infty$, which contradicts to $\bar{h} = 0$. So $T_{\max} = \infty$.

Next we consider the case $0 < \bar{h} < \infty$. In this case, we choose N greater than $\frac{n}{h^-}$ (now h^- is the uniform positive lower bound of $h(t)$ in $[T^*, T_{\max})$). Therefore by (6.2), the inner radius $r_B^-(t)$ and outer radius $r_B^+(t)$ of M_t satisfy the following inequalities, respectively

$$\begin{aligned} r_B^-(t) + \frac{n}{h^-} \log(h^- r_B^-(t) - n) &\geq h^-(t - T^*) + r_{in}(T^*) \\ &\quad + \frac{n}{h^-} \log(h^- r_{in}(T^*) - n), \end{aligned} \quad (6.4)$$

and

$$\begin{aligned} r_B^+(t) + \frac{n}{h^-} \log(h^- r_B^+(t) - n) &\geq h^-(t - T^*) + r_{out}(T^*) \\ &\quad + \frac{n}{h^-} \log(h^- r_{out}(T^*) - n). \end{aligned} \quad (6.5)$$

Lemma 16. *When $t \geq T^*$, the regions enclosed by the hypersurfaces M_t are increasing. Furthermore $T_{\max} = \infty$, and the solutions to (1.1) expand uniformly to ∞ as $t \rightarrow \infty$.*

Proof. For $t \geq T^*$, (6.2) implies that $r_B(t)$ is increasing since $r_B(t) > \frac{n}{h^-}$ initially. By containment principle again, the enclosed regions of M_t are increasing. Moreover, all M_t 's are contained in the regions between $\partial B_{r_B^-(t)}(O)$ and $\partial B_{r_B^+(t)}(O)$ for every $t \in [T^*, T_{\max})$.

Suppose T_{\max} is finite. Integrating Equation (6.2) from T^* to t , we have

$$r_B^+(t) - r_{out}(T^*) = \int_{T^*}^t h(\tau) d\tau - \int_{T^*}^t \frac{n}{r_B^+(\tau)} d\tau,$$

which implies that $r_B^+(t)$ is uniformly bounded from above in $[T^*, T_{\max})$. This is a contradiction to (6.1). Therefore $T_{\max} = \infty$.

Obviously $r(z, t) \rightarrow \infty$ for any $z \in \mathbb{S}^n$ as $t \rightarrow \infty$ by (6.4), (6.5) and the containment principle, which implies that M_t expands to ∞ in this case. The lemma follows. \square

Remark 8. *Lemma 16 and Proposition 1 imply the limit*

$$\lim_{t \rightarrow \infty} \psi(t) = 0.$$

We don't know whether the rescaled mean curvature \tilde{H} is uniformly bounded from above or not, but we can prove that if the rescaled hypersurface \tilde{M}_t converges to a smooth hypersurface, it must be a sphere. To this end, we need to estimate the lower bound of the rescaled mean curvature. Again we consider the function

$$\Phi = \frac{H}{\beta - \mathcal{Z}}$$

for some constant β . As in Lemma 7 we have the evolution equation of Φ

Lemma 17. *For $t \in [0, \infty)$ and $z \in \mathbb{S}^n$,*

$$\begin{aligned} \frac{\partial}{\partial t} \Phi &= g^{ij} \bar{\nabla}_i \bar{\nabla}_j \Phi - \frac{2}{\beta - \mathcal{Z}} g^{ij} \bar{\nabla}_i \Phi \bar{\nabla}_j \mathcal{Z} \\ &\quad + \frac{1}{(\beta - \mathcal{Z})^2} \{ (\beta |A|^2 + h) H - [2H^2 + h(\beta - \mathcal{Z}) |A|^2] \}. \end{aligned}$$

For any $t_0 \in [T^*, \infty)$, let $\beta = 2r_{out}(t_0)$ in Lemma 17. Then by Lemma 16, for any $t \in [T^*, t_0]$,

$$\mathcal{Z} = \langle X, \mathbf{v} \rangle \leq r_{out}(t_0).$$

Applying the maximum principle to the evolution equation of Φ , by the same approach as in the proof of Lemma 8 we have

Lemma 18. *There is a positive constant c_{15} such that for any $(x, \tilde{t}) \in M^n \times [0, \infty)$*

$$\tilde{H}(x, \tilde{t}) \geq c_{15}.$$

At last we show that the eigenvalues of the second fundamental form approach to each other, when $\tilde{t} \rightarrow \tilde{T}_{\max}$. As before we consider the function defined in section 4

$$f = \frac{|A|^2}{H^2}.$$

It is easy to see that f is a scaling invariant. We also have the evolution equation of \tilde{f} as in (4.9). By similar discussion as in the proof of Theorem 1(I), the rescaled evolving hypersurfaces $\tilde{M}_{\tilde{t}}$ tends to a sphere as $\tilde{t} \rightarrow \infty$. This finishes the proof of Theorem 1(III).

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